Solving Differential Equations by Lie Groups

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1 Introduction

In the middle of the nineteenth century the Norwegian mathematician named Sophus Lie pioneered work on the theory of continuous group transformations, now called *Lie groups*. Initially inspired by Galois' use of finite groups to solve algebraic equations, Lie set out to see if continuous groups could solve differential equations. Incredibly, Lie discovered that there is an underlying continuous symmetry group underlying differential equations and that it could be exploited to solve differential equations or reduce their order. Since the history of mathematics leading up to the development of Lie groups is truly fascinating, I implore the reader to consult a more detailed account of the history [1, 6]. I will, however, briefly outline the parts that could provide insight to Lie's brilliant perspective.

The story begins around 1830 with arguably the most romantic figure in the history of mathematics, *Evariste Galois*. I won't go into his limited life story (he died at 21), but I will say that it reads like a Dostoyevsky novel. Before Galois' time, mathematicians such as Lagrange, Gauss, and Abel questioned whether a higher order algebraic equation could be solved by constructing lower-order algebraic equations whose solutions are functions of the original higher order equation. Simply put, could higher-order algebraic equations be solved in terms of the coefficients, algebraic operations, and radicals? Not only did Galois solve the problem, but he determined a general criterion about the solvability of algebraic equations. Galois' discovery, known as *Galois Theory*, introduced the concept of *groups* in that there are symmetries underlying the solutions of algebraic equations in the form of rearranging, or permuting the roots. The first person to write a book about Galois Theory was Camille Jordan who, when writing the book in 1870, acquired two postgraduate students named *Sophus Lie* and *Felix Klein*.

Before working with Jordan, Lie, starting his mathematics career at the age of 26, was enthralled by geometry. Lie loved the work of both Poncelet and Plucker, where geometry was studied as families of lines, or spheres, as opposed to a collection of points. Lie met up and worked on similar geometry problems with Klein, who was a former student of Plucker. Klein was a vital part of Lie's success because unlike Lie, Klein had his doctorate degree with 7 years of experience in the academic community. Working together with Jordan, Lie and Klein studied *W-Curves*, which are homogenous curves invariant under a group. This work was influential in developing Lie groups because it made Lie study one-parameter subgroups of a group of projective transformations. Also, the work led Lie to discover *contact transformations*, which are surface mappings in a space that includes points and their tangents. Although Lie and Klein bonded over their love of geometry, symmetry, and group theory, they split up shortly after.

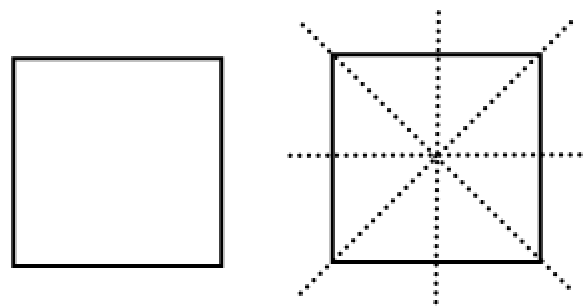
The birth of Lie Groups came about when Lie, as professor in Norway, changed his focus from contact transformations to the study of transformation groups. The motivating question was as follows: *"How can the knowledge of a stability group for a differential equation be utilized towards its integration?."* Significantly, this is where Galois Theory inspired Lie. Drawing directly from Galois' idea of permuting roots of algebraic equations, Lie thought that a point transformation will leave a differential equation stable if it permutes the solutions. This line of thinking gave birth to the theory of Lie Groups. The concepts and theorems that encompass Lie's idea on the symmetry of differential equations will be the topic of this report.

Since I am constrained by length, I am going to avoid some of the mathematical richness within Lie groups and only explain the essentials of Lie groups in their application to differential equations. Since Lie himself developed Lie groups through a geometric view, I will do my best to explain them in such a way. The presented material assumes a background in differential equations and linear algebra. I will begin with an introduction to both symmetry and groups and then develop Lie groups for a 1st order ODE. [1, 2, 6]

2 Symmetries and Groups

What makes something symmetric? Take a square for example. Many people would agree that a square, given its even number of equal length sides, seems symmetric. The square by itself, however, is not symmetric. It's what we *do* to the square that makes it symmetric. We can rotate it 90 degs, flip it over its middle horizontal axis, turn it 180 degs, flip it over its middle vertical axis and the square would look like it never moved. We can look at these movements, independently and all together, of the square as *transformations* that keep the square in it's position.

In this case, we see symmetry as a transformation of the square that preserves its position relative to



this page. The transformations exemplified above, along with others not listed, that keep the square in its exact position together form a *group*. Importantly, in order to make a group, these transformations must all preserve the defined property independently and also when operated together. For example, turning the square by 90 degs preserves the symmetry just as turning the square consecutively by 90 degs and then by 180 degs does. This leads us to the formal definition of a group.

Definition 1 *A group G consists of a set of operations $G = \{g_1, g_2, g_3, ...\}$ called group operations, together with a combination operator, * , called group multiplication such that the following axioms are satisfied [3]:*

(I) Closure: $g_i \in G$ and $g_j \in G$, then $g_i * g_j \in G$

(II) Associativity: For all $g_i \in G, g_j \in G, g_k \in G$, the following must be true, $(g_i * g_j) * g_k = g_i * (g_j * g_k)$

(III) Identity: There is a group operation I, (Identity Operator), with the property that $(g_i * I) = g_i = (I * g_i)$

(IV) Inverse: $(\forall g_i \in G)(\exists g_i^{-1})$ such that $(g_i * g_i^{-1}) = I = (g_i^{-1} * g_i)$

The closure axiom ensures, as we noted before, that for a group every multiplication of group operations has to be itself a group operation. The associativity axiom states that for a group the order of the multiplication of group operations does not matter – turning the square 90 degs and then 180 degs is the same as turning the square 180 and then 90. The identity axiom states that there is a group operation in G that has no effect under group multiplication. The inverse axiom states that for each group operation, the inverse of that group operation takes it back to the original state – turning the square 90 degs clockwise and then turning it 90 counterclockwise.

So how does all of this fit into the context of differential equations? What geometric object is associated with a differential equation and could have a group? Let's find out by exploring a simple first order differential equation given below.

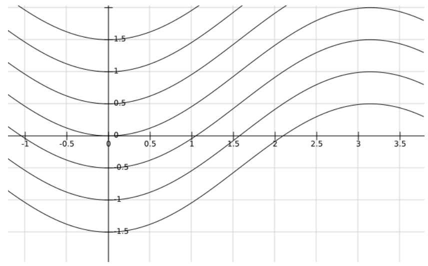
$$y' = sin(x) \tag{1}$$

Solving (1) gives the *general solution* of $y = -cos(x) + c$ where c is the arbitrary integration constant. Importantly, since y' is only dependent on x, it is *invarient* under a shift in the y-direction. This can be seen in plotting y for different values of c as seen in Figure 2a. Lets look at another example below.

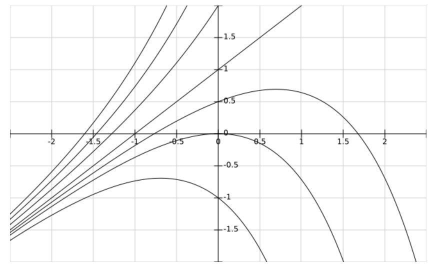
$$y' = y - x \tag{2}$$

Solving (2) gives the general solution of $y = ce^x + x + 1$ where c is the arbitrary constant. What we can gather from this is that all differential equations have a one-parameter family of solutions described by $y = \phi(x, c)$. The family of solutions to (2) can be seen in Figure 2b. In this case, however, the invarient tranformation is not easily seen but it must exist because changing the value of c changes one solution to another.

We can now see that the symmetry of differential equations lies in the invarient transformation from one solution curve to another by the constant of integration. In addition, these translations follow the theory of groups where any number of translations is itself a translation. For our purpose, we will be looking at equations such as (2) where the invarient transformation is not a trivial translation in one direction as in (1). A goal of ours is to develop Lie groups in order to transform an equation such as (2) in such a way that that its transformations become unidirectional translations. Although (1) would never be a analyzed by such symmetry methods, it does help in gathering some of the geometric intuition behind Lie groups.



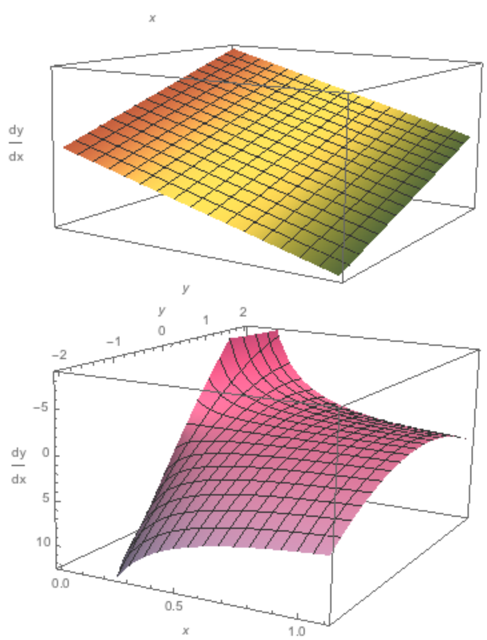
(a) Solution curves for eq (1).



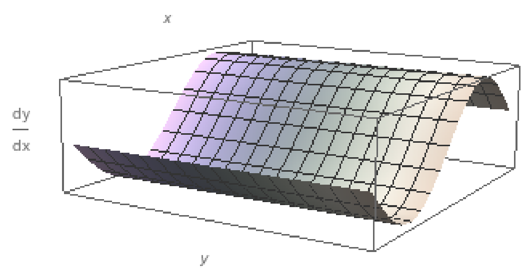
(b) Solution curves for eq (2).

Figure 1: Family of solution curves with varing c values.

To finalize the notion of differential equation symmetries, let's expand the view of the object to 3-dimensions. We can visualize the differential equation as a *surface* in 3-dimensions with coordinates (x, y, y') as shown in Figure 2. Each point on the surface corresponds to a specific solution curve which is identified by its specific value of c. From visualizing the surfaces, we can see that by changing c, a solution moves continuously to other solutions on the surface. Significantly, using group theoretic language, group operations on the surface do not change the defining structure of the surface, which is determined by the differential equation.



(a) Surfaces for (2), top, and $\frac{dy}{dx} = y^2 - \frac{y}{x}$, bottom.



(b) Surface for (1).

Figure 2: ODE surface plots generated by *Mathematica*.

Looking at the differences between the surfaces in Figure 2 also emphasizes a particular effect of symmetry analysis, which happens to be our goal. The goal, as previously stated, is to use Lie group symmetries to change the variables in such a way that the symmetry transformations become translations in one of the variables. Looking at the simple structure of the surface generated by (1) in Figure 2b, we can see that $\frac{dy}{dx}$ is independent of y. The surfaces in Figure 2a on the other hand are dependent on both x and y, leading to a more complicated surface. Importantly, Lie group analysis enables us to change the complex surfaces seen in Figure 2a to much simpler ones such as Figure 2b. Remember, though, that the symmetry operation takes solutions to other solutions on the surface and does not change the surface itself. Instead, the underlying components of the symmetry enable us to find a new coordinate system which simplifies the surface.

3 Lie Groups

Since we have lightly developed the idea of groups and symmetry in the context of differential equations, we are ready to talk directly about Lie groups. Fundamentally, a Lie group symmetry is a continuous transformation that takes each solution into another [4]. As seen in the previous sections, this symmetry is due to one parameter – the constant of integration. Let $\boldsymbol{x} = (x, y)$ and $\boldsymbol{X} = (X, Y)$ be points in the Euclidian plane. Let $P_\lambda : \boldsymbol{x} \longmapsto f(\boldsymbol{x}, \lambda) = \boldsymbol{X}$ be a transformation, depending on the parameter λ in $\mathbb{R}$, that takes points $\boldsymbol{x}$ to $\boldsymbol{X}$ (this is also known as an *action*). We will define a Lie group in this context.

Definition 2 *A One-parameter Lie group (or a Lie point transformation) is a group G with the set of transformations P_λ such that the following conditions are satisfied.*

(I) P_λ *is bijective.*

(II) $P_{\lambda_1} \circ P_{\lambda_2} = P_{\lambda_1+\lambda_2}$

(III) $P_0 = I$

(IV) For each λ_1 there exists a unique $\lambda_1 = -\lambda_2$ such that $P_{\lambda_1} \circ P_{\lambda_2} = P_0 = I$.

(V) f is infinitely differentiable and analytic with respect to λ.

The first condition states that the transformation is one-to-one and onto which means that every element in $\boldsymbol{x}$ is mapped to exactly one element in $\boldsymbol{X}$. Drawing a connection to the definition of a group presented earlier, (II) is analogous to the closure axiom and allows us to call the Lie group *additive*, or *abelian.* (III) is analogous to the identity axiom, and (IV) is analogous to the inverse axiom. The last condition states that the transformation group is continuous, which allows for infinitesimal transformations. In fact, the one-parameter groups were developed by Lie in order to define infinitesimal transformations.

Furthermore, if $\boldsymbol{x} = (x, y)$ and $\boldsymbol{X} = (f(x, y, \lambda), g(x, y, \lambda)) = (X, Y)$, where $P_\lambda : \boldsymbol{x} \longmapsto \boldsymbol{X}$, we can write the one-parameter group in coordinates as a pair of functions of three variables.

$$P_\lambda : \begin{cases} x_\lambda = f(x, y, \lambda) \\ y_\lambda = g(x, y, \lambda) \end{cases} \tag{3}$$

The coordinates, (x_λ, y_λ), represent the image of the coordinates, (x, y), under the transformation P_λ. We can also look at condition (II) in coordinate terms which define the one-parameter group:

$$P_{\lambda_1} \circ P_{\lambda_2} = P_{\lambda_1+\lambda_2} : \begin{cases} f(f(x, y, \lambda_1), g(x, y, \lambda_2)) = f(x, y, \lambda_1 + \lambda_2) \\ g(f(x, y, \lambda_1), g(x, y, \lambda_2)) = g(x, y, \lambda_1 + \lambda_2) \end{cases} \tag{4}$$

Importantly, the set of transformation, P_λ, makes the one-parameter Lie group only with the defining parameterization $\lambda \longmapsto P_\lambda$. With respect to the symmetry of differential equations, the parameter is the constant of integration. We will develop the geometrical intuition of the one-parameter Lie group in the following section. [4]

4 Orbits and Symmetry Condition

A one-parameter group can be visualized through the set of its orbits [5]. An orbit is a continuous curve tracing the path of a point as it changes continuously through solution curves. The continuous change of the point comes from varying the integration constant. Importantly, the orbit provides a direction in which solutions slide into other solutions. As we will show later, utilizing the orbits, we can transform the variables into a new canonical coordinate system in which the solutions of the differential equation slide into each other in only one of the canonical coordinate directions. For a quick example with some hand waving, lets look at the Bernoulli equation:

$$y' = \frac{y(x - y)}{x^2} \tag{5}$$

Transforming (5) into such a coordinate system $(r,s) = (x, \frac{-x}{y})$, by the total derivative, we get $\frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{-y+y'x}{y^2} = \frac{-y+xy(x-y)/x^2}{y^2} = \frac{-y+y-y^2/x}{y^2} = -\frac{1}{x} = -\frac{1}{r}$ where solutions slide into each other in the direction of one of the canonical coordinates. Since $\frac{ds}{dr} = -\frac{1}{r}$ is independent of s, we can easily solve it and substitute the original variables in to get the solution to the Bernoulli equation, $y = \frac{x}{lnx+C}$. Significantly, we can see that the standard change of variables method in solving differential equations is just a special case of Lie group methods.

So how do we get this coordinate system? We start with determining the *symmetry condition* (also known as the invariance condition). Suppose we have a differential equation:

$$\frac{dy}{dx} = h(x,y) \tag{6}$$

that has a one-parameter Lie group symmetry $P_\lambda : (x,y) \longmapsto (X,Y)$. Because the symmetry preserves the solution to the differential equation, we can write (6) as,

$$\frac{dY}{dX} = h(X,Y) \tag{7}$$

Using the coordinate transformation (3), we can solve (7) by the total derivative, using the multivariable chain rule, as we did previously with the Bernoulli equation. Remember that $(X,Y) = (f(x,y,\lambda), g(x,y,\lambda))$. For example, the total derivative of Y with respect to x.

$$D_x Y = \frac{dY}{dx} = \frac{dg}{dx} = \frac{\partial g}{\partial x}\frac{dx}{dx} + \frac{\partial g}{\partial y}\frac{dy}{dx} + \frac{\partial g}{\partial \lambda}\frac{d\lambda}{dx} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = Y_x + y'Y_y$$

The resulting transformation:

$$\frac{dY}{dX} = \frac{D_x Y}{D_x X} = \frac{Y_x + y'Y_y}{X_x + y'X_y} = h(X,Y) \tag{8}$$

Plugging (6) into (8), we get a nonlinear partial differential equation that implicitly describes the symmetry.

$$\frac{Y_x + h(x,y)Y_y}{X_x + h(x,y)X_y} = h(X,Y) \tag{9}$$

Solving for X and Y would give us our canonical coordinates. It is, however, generally impossible to solve for X and Y in this equation because of the nonlinearity as well as the fact that λ needs to vanish.

To think about the symmetry condition in a broader geometrical context, let's again look at the surface plots that each represent a differential equation in Figure 2. Significantly, note that equation (9), in its essence, requires that the symmetry transformation takes a point on the surface to another point on the surface. Thus, we can think of the one-parameter lie group as a group of transformations that leave the surface invariant. For example, take the equation in Figure 2a and rearrange it so the right-hand side is equal to 0, representing $\frac{dy}{dx}$ by p . We can represent the left-hand side as a function of x, y, p.

$$y^2 - \frac{y}{x} - p = 0$$
$$F(x,y,p) = 0$$

The surface equation must be invariant, or unchanged, under the one-parameter lie group transformation, $P_\lambda : (x,y) \longmapsto (X,Y)$, so continuing from the above equations , with $p' = \frac{dY}{dX}$, we get:

$$F(x,y,p) = F(X,Y,p') = 0$$

We can now express (9) more generally.

Definition 3 *A family of curves represented by a surface, F, is said to be invariant under the Lie group $\boldsymbol{G}$ with the transformation $P_\lambda : \boldsymbol{x} \longmapsto f(\boldsymbol{x},\lambda) = \boldsymbol{X}$ if and only if:*

$$F(\boldsymbol{x},p) = F(f(\boldsymbol{x},p,\lambda)) = F(\boldsymbol{X},p') = 0 \tag{10}$$

Eq (10) states the same thing as eq (9). We will utilize the surface equation in the next section, but what we should see now is that the surface is composed of curves that satisfy (6) and (7). Unfortunately, the presented invariance condition is not enough because it is generally impossible to solve for X and Y. We can, however, linearize the condition in as formulated by (9) into something we can usually solve. [3, 4, 6]

5 Linearized Symmetry Condition

Now that we have an equation (9) describing the orbits of points on the solution curves, we would like to linearize it in order to solve for the canonical coordinates. To do so, let's imagine a vector field on the surface such that the vectors are tangent to the orbits on the surface. For the one-parameter lie group symmetry, $P_\lambda : (x, y) \longmapsto (X, Y) = (f(x, y, \lambda), g(x, y, \lambda))$, we can derive the tangents to the orbits, ξ and η, which are known as *infinitesimal transformations*.

$$\frac{dX}{d\lambda} = \frac{df}{d\lambda} = \xi(X, Y) \quad \text{where} \quad \left.\frac{dX}{d\lambda}\right|_{\lambda_o} = \xi(x, y),$$
$$\frac{dY}{d\lambda} = \frac{dg}{d\lambda} = \eta(X, Y) \quad \text{where} \quad \left.\frac{dY}{d\lambda}\right|_{\lambda_o} = \eta(x, y)$$

Note that $\lambda_o = 0$ is the identity element where $P_{\lambda_o} : \boldsymbol{x} \longmapsto f(\boldsymbol{x}, \lambda_o) = \boldsymbol{x}$. The ξ and η functions are tangents to the coordinate curves that we are trying to determine. Thus, the inifinitesimal transformations are also called the *vector field* of the one-parameter lie group. Furthermore, we will use these functions within the linearized symmetry condition. In order to linearize the symmetry condition (9), we must expand X, Y, and $h(X, Y)$ in taylor series around λ_o.

$$\begin{aligned}
X &= x + (\lambda + \lambda_o) \left.\frac{dX}{d\lambda}\right|_{\lambda_o} + \mathcal{O}((\lambda + \lambda_o)^2) = x + (\lambda + \lambda_o)\xi(x, y), \\
Y &= y + (\lambda + \lambda_o) \left.\frac{dY}{d\lambda}\right|_{\lambda_o} + \mathcal{O}((\lambda + \lambda_o)^2) = y + (\lambda + \lambda_o)\eta(x, y), \\
h(X, Y) &= h(x, y) + (\lambda + \lambda_o)(h_x(x, y) \left.\frac{dX}{d\lambda}\right|_{\lambda_o} + h_y(x, y) \left.\frac{dY}{d\lambda}\right|_{\lambda_o}) + \mathcal{O}((\lambda + \lambda_o)^2) \\
&= h(x, y) + (\lambda + \lambda_o)(h_x(x, y)\xi(x, y) + h_y(x, y)\eta(x, y)) \qquad (11)
\end{aligned}$$

The final linearized results for each variable taylor expansions comes from ignoring the higher order terms of $\mathcal{O}((\lambda + \lambda_o)^2)$ and higher. Now we can determine the components of (9) in linear terms.

$$\begin{aligned}
Y_x &= (\lambda + \lambda_o)\eta_x \\
Y_y &= 1 + (\lambda + \lambda_o)\eta_y \\
X_x &= 1 + (\lambda + \lambda_o)\xi_x \\
X_y &= (\lambda + \lambda_o)\xi_y
\end{aligned}$$

When substituted back into the left and right hand side of (9):

$$\begin{aligned}
\frac{Y_x + h(x, y)Y_y}{X_x + h(x, y)X_y} &= \frac{(\lambda + \lambda_o)\eta_x + h * (1 + (\lambda + \lambda_o)\eta_y)}{1 + (\lambda + \lambda_o)\xi_x + h * ((\lambda + \lambda_o)\xi_y)} = \frac{h + (\lambda + \lambda_o)(\eta_x + h\eta_y)}{1 + (\lambda + \lambda_o)(\xi_x + h\xi_y)} \\
&= h(X, Y) = h(x, y) + (\lambda + \lambda_o)(h_x\xi + h_y\eta), \\
\frac{h + (\lambda + \lambda_o)(\eta_x + h\eta_y)}{1 + (\lambda + \lambda_o)(\xi_x + h\xi_y)} &= h(x, y) + (\lambda + \lambda_o)(h_x\xi + h_y\eta)
\end{aligned}$$

Simplifying the expression by multiplying both sides and again ignoring higher order terms, we recieve the resulting equation:

$$\eta_x - \xi_y h^2 + (\eta_y - \xi_x)h - (\xi h_x + \eta h_y) = 0 \qquad (12)$$

This result represents the linearized symmetry condition for first order ODEs in two dimensions. In the literature [1,2,4], this equation is also known as the *first prolongation* formula which essentially relates the infinitesimal transformation of $h(x, y)$ to η and ξ.

Just as we did in the previous section, let's advance our understanding by looking at the surface equation for the differential equation in the context of the linearized symmetry condition. Before we do so, we should understand the invariance of a function. Let's pretend that the function $h(x, y)$ is invariant, which means that the transformations move points on the curve $h(x, y)$ to other points on the same curve. If this was true, then $h(x, y) = h(X, Y)$. Looking at eq (11), we see that $h(x, y) = h(X, Y)$ holds true only if $h_x(x, y)\xi(x, y) + h_y(x, y)\eta(x, y) = 0$. We can rewrite this linearized symmetry condition for a curve as follows:

$$\mathbf{\Gamma} h(x, y) = 0 \quad \text{where} \quad \mathbf{\Gamma} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$$

$\mathbf{\Gamma} h(x, y)$ is known as the *Lie derivative* of h, and $\mathbf{\Gamma}$ is an operator known as the *infinitesimal generator* of the one-parameter lie group that leaves the curve invariant. Since we are dealing with differential equations, we need to look at a family of curves or surface, $F(x, y, p)$, where $p = \frac{dy}{dx} = h(x, y)$. Using eq (11), with $\gamma = h_x\xi + h_y\eta$, we linearize the surface equation under the one-parameter Lie group and determine the constraint.

$$F(X, Y, p') = F(x, y, p) + (\lambda + \lambda_o)(F_x(x, y, p)\xi(x, y, p) + F_y(x, y, p)\eta(x, y) + F_p(x, y, p)\gamma(x, y, p))$$

which satisfies Definition 3 only if,

$$F_x\xi + F_y\eta + F_p\gamma = 0$$

We can now write the linearized invariance condition in terms of the Lie derivative and infinitesimal generator.

$$\mathbf{\Gamma} F(x, y, p) = 0 \quad \text{where} \quad \mathbf{\Gamma} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial p} \tag{13}$$

Since differential equations, exemplified in Figure 2, have the form $h(x, y) - p = 0$, we can simplify (13) by making $F_p = -1$. In addition, from eq (12), we can represent γ in terms of h, η ,and ξ. Together, equations (12) and (13) are known as the *determining equations.*

The linearized symmetry condition was a great advancement by Sophus Lie. By replacing the finite nonlinear invariance condition with an inifinitesimal linear condition, maintaining the symmetry, Lie theory ensures that no matter the nonlinearity of an equation, we will be dealing with a linear invariance condition [6]. Although (12) is vastly more useful than (9), it is still extremely difficult to solve for the infinitesimals, η and ξ. There are, however, complicated algorithms for finding the invariance group that utilize the determining equations. Generally, for first-order ODEs, as in our case, educated guesses of the infinitesimals in equation (12) will suffice. Once the infinitesimal transformations from (12) are determined, we can find a simplifying coordinate system. [3, 4, 6]

6 Canonical Coordinates

The most elegant results from Lie groups come by using canonical coordinates. Significantly, the symmetry transformations in the original coordinate system become one-variable translations in the canonical coordinates – reducing the order of the differential equation. Lets assume that we have determined the inifinitesimal symmetry of the differential equation such that the changes in the parameter λ, or integration factor, are translations in the new canonical coordinate system, written as $(r(x, y), s(x, y))$. Also, the translation could either occur in r or s so we will just pick one, s, for ease. Formally, $P_\lambda : (r, s) \longmapsto (R, S) = (r, s + \lambda)$. To determine the canonical coordinates, we must now mathematically describe the translational behavior of the canonical coordinate system by the infinitesimal transformations. To do so, we find the tangent vectors of the canonical coordinates at the parameter identity element, λ_o.

$$\left.\frac{dR}{d\lambda}\right|_{\lambda_o} = \left.\frac{dR}{dx}\frac{dx}{d\lambda}\right|_{\lambda_o} + \left.\frac{dR}{dy}\frac{dy}{d\lambda}\right|_{\lambda_o} = \frac{dr}{dx}\xi + \frac{dr}{dy}\eta$$

$$\left.\frac{dS}{d\lambda}\right|_{\lambda_o} = \left.\frac{dS}{dx}\frac{dx}{d\lambda}\right|_{\lambda_o} + \left.\frac{dS}{dy}\frac{dy}{d\lambda}\right|_{\lambda_o} = \frac{ds}{dx}\xi + \frac{ds}{dy}\eta$$

Note that $\frac{dR}{dx} = \frac{dr}{dx}$ because we are evaluating it at the identity element. Also, since changing λ results in a translation of the s coordinate, we know that $\frac{dR}{d\lambda}\big|_{\lambda_o} = 0$ and $\frac{dS}{d\lambda}\big|_{\lambda_o} = 1$. Thus, we can rewrite the above equations as follows:

$$\frac{dr}{dx}\xi + \frac{dr}{dy}\eta = 0, \qquad \frac{ds}{dx}\xi + \frac{ds}{dy}\eta = 1 \tag{14}$$

We can similarly see (14) in terms of the infinitesimal operator $\mathbf{\Gamma}$ as $\mathbf{\Gamma}r = 0$ and $\mathbf{\Gamma}s = 1$. The equations in (14) are both first-order linear partial differential equations. To solve for $r(x, y)$ and $s(x, y)$, we will use the *method of characteristics*. For example, take the following linear partial differential equation:

$$a(x, y, z)\frac{\partial z}{\partial x} + b(x, y, z)\frac{\partial z}{\partial y} = c(x, y, z)$$

Let's say that we know the solution of $z(x, y)$ which we can represent as a surface in coordinates x, y, z. Then we know that the normal vector to this surface is:

$$(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1)$$

Its vector field, which must be tangent at every point on the surface $z(x, y)$, must be (a, b, c) because $(a, b, c) \cdot (\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1) = 0$. Therefore, the solution surface of z must be composed of the integral curves of the vector field known as the *characterstic curves*. If we think of these curves as parameterized functions of some variable, t, we can can look at the tangent vector instead as a function $((a(x(t), y(t)), (b(x(t), y(t)), (c(x(t), y(t))$. This leads us to write down the *characteristic equations*:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{c} \tag{15}$$

The concepts underlying the method of characterstics must feel very familiar at this point. Similarly, we have said that (η, ξ) represents the vector field which is tangent to the orbits on the solution surface of the differential equation. The integral curves of (η, ξ) will give us the solutions to r and s – the canonical coordinates. We can now rewrite equation (14) as follows with ds.

$$\frac{dx}{\xi} = \frac{dy}{\eta}, \qquad \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{ds}{1} \tag{16}$$

To solve for r, we find $r = C = f(x, y)$ such that, the left equation in (16) is satisfied.

$$\eta dx = \xi dy$$
$$\int \eta dx = \int \xi dy$$
$$N(x, y) - E(x, y) = C$$

Where $N(x, y) = -c_1 + \int \eta dx$ and $E(x, y) = -c_2 + \int \xi dy$ with the constants of integration of $\int \eta dx$ and $\int \xi dy$, c_1 and c_2, respectively, collected on the right in C. r has to be C because of equation (14). We can solve for s from the right side of equation (16).

$$s = \int \frac{dx}{\xi} = \int \frac{dy}{\eta} = \int ds$$

Now that we know the canonical coordinates, $(r(x, y), s(x, y))$, we can determine the expression of the differential equation in terms of the canonical variables using the symmetry condition in equation (8).

$$\frac{ds}{dr} = \frac{D_x s}{D_x r} = \frac{s_x + h(x, y)s_y}{r_x + h(x, y)r_y} \tag{17}$$

Since equation (17) will express the differential equation in terms of the old variables, x and y, we need an inverse relationship, $x = x(r, s)$ and $y = y(r, s)$, to have equation (17) in its simplified form. We can do this by solving the system of equations, $r(x, y)$ and $s(x, y)$, for x and y. If everything is done correctly, (17) will depend on only r. The solution of (17) in terms of the original variables will be the final solution. [2, 3, 4, 6]

7 Example

Let's solve the equation presented in Figure 2a with the method of Lie groups.

$$\frac{dy}{dx} = y^2 - \frac{y}{x} \tag{18}$$

The first step is to determine the linearized invariance condition for Equation (18) using Equation (12). We get the following condition with $h = y^2 - \frac{y}{x}$.

$$\eta_x - \xi_y(y^2 - \frac{y}{x})^2 + (\eta_y - \xi_x)(y^2 - \frac{y}{x}) - (\xi(\frac{y}{x^2}) + \eta(2y - \frac{1}{x}) = 0$$

Multiplying by x, simplifying, and collecting terms.

$$\eta_x x - \xi_y(xy^4 - 2y^3 + \frac{y^2}{x}) + (\eta_y - \xi_x)(xy^2 - y) - \xi(\frac{y}{x}) - \eta(2yx - 1) = 0 \tag{19}$$

In guessing a value for η and ξ, it's best to make some simplifying assumptions along with trying simple polynomials in a single variable. Let's assume that they depend only on one of the variables so that $\xi = f_1(x)$ and $\eta = f_2(y)$. This assumption makes η_x and ξ_y equal to 0. Equation (19) simplifies to,

$$\eta_y(xy^2 - y) - \eta(2yx - 1) = \xi_x(xy^2 - y) + \xi(\frac{y}{x})$$

Again, let's assume that η and ξ are polynomials in a single variable such that η_y and ξ_x become -1 and 1. This makes $\xi = x$ and $\eta = -y$. Trying it out in the above equation show that they work, so we will stop and use them for the inifitesimal transformations. In most situations, it is much simpler to use an already developed computer algorithm that computes the infinitesimal vectors. Now that we know η and ξ, the next step is to determine the canonical coordinates, $r(x, y)$ and $s(x, y)$, using equation (14).

$$x\frac{dr}{dx} - y\frac{dr}{dy} = 0, \qquad x\frac{ds}{dx} - y\frac{ds}{dy} = 1 \tag{20}$$

The characteritic equation follows,

$$\frac{dx}{x} = \frac{dy}{-y}, \qquad \frac{dx}{x} = \frac{dy}{-y} = \frac{ds}{1} \tag{21}$$

Solving for r in (21), we get,

$$\int ydx = \int -xdy \rightarrow yx + yx = c \rightarrow xy = \frac{c}{2} \rightarrow C = xy = r$$

Solving for s in (21), we get,

$$s = \int ds = \int \frac{dy}{-y} = -ln(y)$$

Since it doesn't matter if $\mathbf{\Gamma}s = 1$ or $\mathbf{\Gamma}s = -1$, we will drop the negative sign and use the solution, $s = ln(y)$. Figure 3 nicely illustrates the surface transformation of (18) in canonical coordinates. Now that we have the canonical coordinates, the next step is to use the symmetry condition in equation (17) to have the differential equation transformed to canonical coordinates.

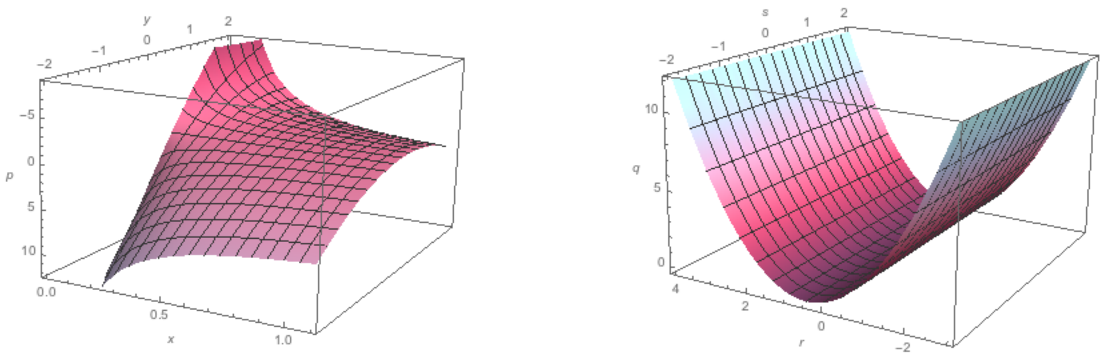
$$\frac{ds}{dr} = \frac{s_x + h(x,y)s_y}{r_x + h(x,y)r_y} = \frac{(y^2 - \frac{y}{x})(\frac{1}{y})}{y + (y^2 - \frac{y}{x})x} = \frac{y - \frac{1}{x}}{xy^2} \tag{22}$$

Using the inverse relationship between the original and canonical coordinates, $x = e^s$ and $y = re^{-s}$, we can get equation (22) in only canonical coordinates.

$$\frac{ds}{dr} = \frac{re^{-s} - e^{-s}}{r^2e^{-s}} = \frac{r-1}{r^2} = \frac{1}{r} - \frac{1}{r^2} \tag{23}$$

We have now transformed equation (18), which is dependent on both x and y, to equation (23), which is dependent on only one of the variables. This result exemplifies the elegance of lie group symmetries. Solving equation (23), we get $s = ln(r) + \frac{1}{r} + c$. Using the inverse relationship of the coordinates, we arrive at the general solution of (18) in terms of the original variables. [3, 4]

$$ln(y) = ln(xy) + \frac{1}{xy} + c \longrightarrow y = xye^{\frac{1}{xy}}e^c \longrightarrow \frac{e^{-c}}{x} = e^{\frac{1}{xy}}$$
$$\longrightarrow -c - ln(x) = \frac{1}{xy} \longrightarrow y = -\frac{1}{x(c + ln(x))}$$



(a) $F(x, y, p) = y^2 - \frac{y}{x} - p = 0$

(b) $F(r, s, q) = r^2 - r - q = 0$

Figure 3: Comparison of surface plots between original and canonical coordinates. $F(x, y, p)$ was transformed to $F(s, r, q)$ using the inverse coordinate transformations. q is equation (18) with $x = e^s$ and $y = re^{-s}$ plugged in. Importantly, the shape of (b) is independent of s.

8 Conclusion

Although the presented material develops lie groups in the case of first order differential equations, the general ideas are the same for higher order ODEs and partial differential equations. Furthermore, the mathematics of Lie groups is incredibly rich, so my report barely scratches the surface. If more pages were permitted, or if I was more efficient, I would have presented the idea of a *Lie algebra* as well as the idea of a *manifold*. Given the length, I believe this paper provides a basic introduction to symmetry analysis by Lie groups.

Personally, as a student interested in understanding the complexity of biology, I found studying Lie groups to be an enlightening experience. Fundamentally, Lie groups provide insight to natural phenomenon by exploring the invariance of their corresponding mathematical representations. We can even look at physical laws as merely consequences of the symmetry properties that govern the system. Thus, symmetries are as fundamental as the physical laws themselves. Perhaps, the apparent asymmetry of biological phenomenon could be uncovered by studying their underlying symmetries... or at the very how these asymmetries c. I'll end the report with the following quote, [6]

> *"In Greek mythology, Ariadne, the daughter of Pasiphae and Minos, the king of Crete, gives the hero Theseus a thread whereby he is able to mark the way of his escape from the labyrinth. In physics, symmetries provide the Ariadne thread that enables us to navigate our way through the infinitely varied and complex labyrinth of natural phenomena."*
>
> — B. Cantwell

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